Math 522 Exam 12 Solutions

1. Let $n \in \mathbb{N}$. Suppose we have players $P_1, P_2, \ldots, P_n$, with additive valuation functions $f_1, f_2, \ldots, f_n$. Let $B_1, B_2, \ldots, B_n$ be bundles of goods and/or payments. Suppose that we assign bundle B_i to player P_i (for $1 \leq i \leq n$). Prove that if this assignment is envy-free then it is fair.

   Let i be arbitrary. Because the assignment is envy-free, then $f_i(B_j) \leq f_i(B_i)$ for all j. Summing over all j, we get $f_i(B_1)+f_i(B_2)+\cdots+f_i(B_n) \leq nf_i(B_i)$. Since f_i is additive, we have $f_i(B_1+B_2+\cdots+B_n) \leq nf_i(B_i)$. Dividing both sides by n, we conclude that player i thinks this assignment is fair. Since i was arbitrary, all the players think the same.

2. Suppose players A, B, C go in together on a fruit basket costing \$7.50. They want to divide the three fruits therein so that each gets one fruit. Use the HRS algorithm to find an envy-free assignment, dividing any surplus evenly. Their valuations are as follows.

	A	B	C
apple	2.00	3.50	2.50
pear	2.00	4.00	4.00
grapes	4.00	5.50	6.00

   Let's begin with the efficient assignment where A gets the apple (and pays \$2), B gets the pear (and pays \$4), and C gets the grapes (and pays \$6). [if we were unlucky and started with an inefficient assignment, we would need to rearrange at some point]

   Our initial envy matrix is Fig.1 below. We see that B and C each envy A, so we give B \$1.50 and give C \$0.50 in the first round. The result is Fig.2 below. Now C envies B, so we give C \$1.00 in the second round. The result is Fig.3 below. Now there is no envy, and \$1.50 surplus. Dividing this evenly, we get a final outcome of A paying \$1.50, B paying \$2.00, C paying \$4.00.

1	A	B	C
A	0	-2.00	-2.00
B	1.50	0	-0.50
C	0.50	0	0

2	A	B	C
A	0	-0.50	-1.50
B	1.50	1.50	0
C	0.50	1.50	0.50

3	A	B	C
A	0	-0.50	-0.50
B	1.50	1.50	1.00
C	0.50	1.50	1.50